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Solving System of Fuzzy Linear Equations in Matrix Form Method.

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Abstract— With uncertainty on the parameters, a linear system of equations plays an important role in Economics and Finance. In Economics, linear systems of equations with uncertainty on parameters are widely used due to some imprecise data on the relation of a linear system of equations. In this paper, a detailed study of solution technique of system of fuzzy linear equations has been done. Appropriate applications have been also given to illustrate the technique. Before starting the main discussion, we have reviewed basic definitions and results of the fuzzy set theory. Then we have discussed the method to solve the system of fuzzy linear equations. Finally, we have presented some applications of the method for three different types of example.

Index Terms—Fuzzy set, Characteristic function, Fuzzy power set, $\alpha -$ cuts, Strong $\alpha -$ cuts, Fuzzy linear system, Crisp matrix, Fuzzy matrix,

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1 Introduction

SYSTEM of linear equations plays a major role in various areas such as operational research, physics, statistics, engineering, and social sciences. In many applications, the parameters of linear equation systems are not always exactly known and stable. This imprecision may follow from the lack of exact information, changeable economic conditions, etc. A frequently used way of expressing the imprecision is to use the fuzzy numbers rather than the crisp numbers. It enables us to consider tolerances for parameters of linear equation systems (the entries of the coefficient matrix and right-hand side vector) in a more natural and direct way. Therefore, fuzzy linear equation systems seem to be more realistic and reliable than the crisp case.

In 1965, **Lotfi A. Zadeh [1]**, professor of electrical engineering at the **University of California (Berkley)**, published the first of his paper on his new theory of ***Fuzzy Sets and System***. After the development of fuzzy set theory researchers have successfully applied in economics**. Buckley** (1987) applied fuzzy mathematics in finance; in 1992 **Buckley** devised a technique to solve fuzzy equations in economics and finance, **Dimitrovski and Matos** (2000) applied Fuzzy set in engineering economic analysis**, Kahraman** (2001) made comparison study on Fuzzy versus probabilistic benefit/cost ratio analysis for public work projects. **Kahraman et.al.** (2000) studied Justification of manufacturing technologies using fuzzy benefit/cost ratio analysis.

The system of linear equations plays a vital role in real life problems such as optimization, economics, and engineering. Due to the presence of uncertainty/ imprecision in the variables of systems of linear equations variables may be considered as fuzzy numbers.

In **Chapter Two**, we review some basic definitions, examples, theorems, results of the fuzzy set theory. In **Chapter Three**, we discussed the method to solve the system of fuzzy linear equation. In **Chapter Four**, we discuss different applications of the methods.

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2 DEFINITIONS

2.1 Fuzzy Set [2][3]

A ***fuzzy set*** is a class of objects with a continuum of the grade of membership. Let X be a space of points. A ***fuzzy set*** A in X is characterized by a membership function $\mu_A : X \to [0,1]$ which associates with each point in X a real number $\mu_A(x)$in the interval $[0,1]$ with the value of μ_A at x representing the grade of membership of $x$in A. Thus the nearer to the value of μ_A unity, the higher the grade of membership of x in A.

2.1 (a) Example

Suppose,$Y: X \to I$ defines a fuzzy set "Youth", where X is the set of ages of human beings defined as follows:

$$Y(x) = \begin{cases} 0 & When\ x \le 10 \\ \frac{x-10}{5} & when\ 10 < x < 15 \\ 1 & when\ 15 \le x \le 35 \\ \frac{50-x}{15} & when\ 35 < x < 50 \\ 0 & when\ x \ge 50 \end{cases}$$

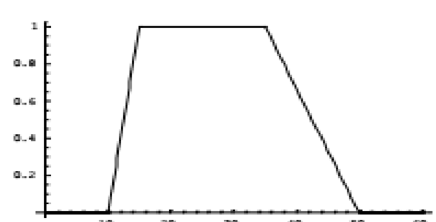

Fig. 1. The membership function of the fuzzy set "Youth".

2.2 Characteristic Function

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A **characteristic function** is a function that declares which elements of the universal X are members of a given set A and which are not. That is, the characteristic function maps elements of X the elements of the set $\{0,1\}$, which is expressed by

$$\mu_A : X \rightarrow \{0,1\}$$

Where X is a universal set.

A ***crisp set*** A can be defined by the characteristic function as follows:

$$\mu_A(x) = \begin{cases} 0\,, & for\ x \notin A \\ 1\,, & for\ x \in A \end{cases}$$

2.2 (a) Example

The simplest example of a crisp set is $A = \{x \in \mathbb{R}: 5 \leq x \leq 10\}$ which can be represented by the following characteristic function

$$\mu_A(x) = \begin{cases} 1\,, & for\ 5 \leq x \leq 10 \\ 0\,, & otherwise \end{cases}$$

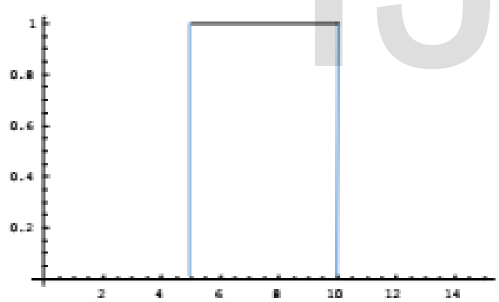

Fig. 2. Graph of the characteristic function of A.

2.3 Fuzzy power set [3][5]

Suppose that X is a universal crisp set, then the class of all fuzzy set defined on X is called the ***fuzzy power set*** and denoted by I^X .

2.4 α-cut [3]

Let $A \in I^X$ is a fuzzy set. Then for every $\alpha \in [0,1]$, A^{α} is called the **α-cut** of A and is denoted by
$A^{\alpha} = \{x \in X : \mu_A(x) \geq \alpha\}$

2.5 Strong $\alpha-$cut [3]

Let $A \in I^X$ is a fuzzy set. Then for every $\alpha \in [0,1]$, $A^{\alpha+}$ is called the **α-cut** of A and is denoted by
$A^{\alpha+} = \{x \in X : \mu_A(x) > \alpha\}$

2.3 (a) Example

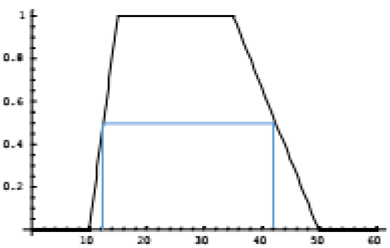

Fig. 1. α-cut and strong α-cut of the fuzzy set "Youth".

3 METHOD

3.1 Solution of Fuzzy Matrix Equation System [6][7][8]

We consider a fuzzy linear system in the form $AX = B$, where A is a $n \times n$ crisp matrix, X is an unknown $n \times 1$ vector of fuzzy numbers $\bar{x}_i$ by which the equation is satisfied and B is a $n \times 1$ fuzzy matrix. Now we can write the corresponding system as follows,

$$\begin{pmatrix} a_{11} & a_{12} & \cdots\cdots\ldots a_{1n} \\ a_{21} & a_{22} & \cdots\cdots\ldots a_{2n} \\ \cdots & \cdots & \ldots\cdots\cdots\ \cdots \\ \cdots & \cdots & \ldots\cdots\cdots\ \cdots \\ a_{n1} & a_{n2} & \cdots\cdots\ldots a_{nn} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ b_n \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots(1.1)$$

For all
$a_{ij} \in \mathbb{R}, 1 \leq i,j \leq n;\ \bar{x}_i = \left(\underline{x_i}, \overline{x_i}\right), 1 \leq i \leq n;\ \bar{b}_i = \left(\underline{b_i}, \overline{b_i}\right), 1 \leq i \leq n.$

Now we can rewrite the system

$$\begin{bmatrix} a_{11L}(\alpha) & a_{12L}(\alpha) & a_{13L}(\alpha) & 0 & 0 & 0 \\ a_{21L}(\alpha) & a_{22L}(\alpha) & a_{23L}(\alpha) & 0 & 0 & 0 \\ a_{31L}(\alpha) & a_{32L}(\alpha) & a_{33L}(\alpha) & 0 & 0 & 0 \\ 0 & 0 & 0 & a_{11U}(\alpha) & a_{12U}(\alpha) & a_{13U}(\alpha) \\ 0 & 0 & 0 & a_{21U}(\alpha) & a_{22U}(\alpha) & a_{23U}(\alpha) \\ 0 & 0 & 0 & a_{31U}(\alpha) & a_{32U}(\alpha) & a_{33U}(\alpha) \end{bmatrix} \cdot \begin{bmatrix} x_{1L}(\alpha) \\ x_{2L}(\alpha) \\ x_{3L}(\alpha) \\ x_{1U}(\alpha) \\ x_{2U}(\alpha) \\ x_{3U}(\alpha) \end{bmatrix} = \begin{bmatrix} b_{1L}(\alpha) \\ b_{2L}(\alpha) \\ b_{3L}(\alpha) \\ b_{1U}(\alpha) \\ b_{2U}(\alpha) \\ b_{3U}(\alpha) \end{bmatrix}$$

like $S.X = Y$ where S is a $2n \times 2n$ crisp matrix and

$$X = \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ \cdot \\ \cdot \\ \underline{x_n} \\ \overline{x_1} \\ \overline{x_2} \\ \cdot \\ \cdot \\ \overline{x_n} \end{pmatrix} \text{ and } Y = \begin{pmatrix} \underline{y_1} \\ \underline{y_2} \\ \cdot \\ \cdot \\ \underline{y_n} \\ \overline{y_1} \\ \overline{y_2} \\ \cdot \\ \cdot \\ \overline{y_n} \end{pmatrix}$$

The matrix S is constructed as follows

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If $a_{ij} \geq 0$ then $S_{i,j} = a_{ij}$ and $S_{i+n,j+n} = a_{ij}$
If $a_{ij} < 0$ then $S_{i,j+n} = a_{ij}$ and $S_{i+n,j} = a_{ij}$
And $S_{i,j}$ which is not determined by (5.2.4.1) is zero.

So S implies,

$$S = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \quad \text{for } 1 \leq i,\ j \leq 2n.$$

$$\therefore \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ \cdot \\ \cdot \\ \underline{x_n} \\ \overline{x_1} \\ \overline{x_2} \\ \cdot \\ \cdot \\ \overline{x_n} \end{pmatrix} = \begin{pmatrix} \underline{y_1} \\ \underline{y_2} \\ \cdot \\ \cdot \\ \underline{y_n} \\ \overline{y_1} \\ \overline{y_2} \\ \cdot \\ \cdot \\ \overline{y_n} \end{pmatrix}$$

We consider $y_i(r), y_l(r)$ and consequently $\underline{x_i}(r), \overline{x_l}(r)$ are all linear function of r where $0 \leq r \leq 1$. It should be noted that the determinant of the coefficients matrix S must be nonsingular and invertible. We find the inverse of S.

$$\Rightarrow \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ \cdot \\ \cdot \\ \underline{x_n} \\ \overline{x_1} \\ \overline{x_2} \\ \cdot \\ \cdot \\ \overline{x_n} \end{pmatrix} = \begin{pmatrix} S_1 & S_2 \\ S_2 & S_1 \end{pmatrix}^{-1} \begin{pmatrix} \underline{y_1} \\ \underline{y_2} \\ \cdot \\ \cdot \\ \underline{y_n} \\ \overline{y_1} \\ \overline{y_2} \\ \cdot \\ \cdot \\ \overline{y_n} \end{pmatrix}$$

$$\therefore \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ \cdot \\ \cdot \\ \underline{x_n} \\ \overline{x_1} \\ \overline{x_2} \\ \cdot \\ \cdot \\ \overline{x_n} \end{pmatrix} = S^{-1} \begin{pmatrix} \underline{y_1} \\ \underline{y_2} \\ \cdot \\ \cdot \\ \underline{y_n} \\ \overline{y_1} \\ \overline{y_2} \\ \cdot \\ \cdot \\ \overline{y_n} \end{pmatrix}$$

This is the fuzzy solution of (1.1).

3.2 Solution of Fully Fuzzy Matrix Equation System [9]

We consider a fully fuzzy linear system in the form $AX = B$ where A is a $n \times n$ fuzzy matrix, x is an unknown $n \times 1$ vector of fuzzy numbers $\bar{x}_i$ by which the equation is satisfied and B is a $n \times 1$ fuzzy matrix. Now we can write the corresponding $n \times n$ system as follows,

$$\begin{pmatrix} \bar{a}_{11} & \bar{a}_{12} & \cdots\cdots & \bar{a}_{1n} \\ \bar{a}_{21} & \bar{a}_{22} & \cdots\cdots & \bar{a}_{2n} \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \bar{a}_{n1} & \bar{a}_{n2} & \cdots\cdots & \bar{a}_{nn} \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} \bar{b}_1 \\ \bar{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \bar{b}_n \end{pmatrix}, \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

For all

$$a_{ij} \in \mathbb{R}, 1 \leq i, j \leq n;\ \bar{x}_i = \left[\underline{x_i}, \overline{x_i}\right], 1 \leq i \leq n;\ \bar{b}_i = \left[\underline{b_i}, \overline{b_i}\right], 1 \leq i \leq n.$$

Converting fuzzy numbers into intervals using the α-cuts, the system (2.1) can be written as,

$$\begin{pmatrix} [\underline{a_{11}}, \overline{a_{11}}] & [\underline{a_{12}}, \overline{a_{12}}] & \cdots\cdots & [\underline{a_{1n}}, \overline{a_{1n}}] \\ [\underline{a_{21}}, \overline{a_{21}}] & [\underline{a_{22}}, \overline{a_{22}}] & \cdots\cdots & [\underline{a_{2n}}, \overline{a_{2n}}] \\ \cdots & \cdots & \cdots\cdots & \cdots \\ \cdots & \cdots & \cdots\cdots & \cdots \\ [\underline{a_{n1}}, \overline{a_{n1}}] & [\underline{a_{n2}}, \overline{a_{n2}}] & \cdots\cdots & [\underline{a_{nn}}, \overline{a_{nn}}] \end{pmatrix} \begin{pmatrix} [\underline{x_1}, \bar{x}_1] \\ [\underline{x_2}, \bar{x}_2] \\ \cdot \\ \cdot \\ \cdot \\ [\underline{x_n}, \bar{x}_n] \end{pmatrix} = \begin{pmatrix} [\underline{b_1}, \bar{b}_1] \\ [\underline{b_2}, \bar{b}_2] \\ \cdot \\ \cdot \\ \cdot \\ [\underline{b_n}, \bar{b}_n] \end{pmatrix} \quad \ldots\ldots\ldots(1.3)$$

System (1.3) can be written in equation form,

$$[\underline{a_{11}}, \overline{a_{11}}]\,[\underline{x_1}, \bar{x}_1] + [\underline{a_{12}}, \overline{a_{12}}]\,[\underline{x_2}, \bar{x}_2] + \ldots\ldots + [\underline{a_{1n}}, \overline{a_{1n}}]\,[\underline{x_n}, \bar{x}_n] = [\underline{b_1}, \bar{b}_1]$$
$$[\underline{a_{21}}, \overline{a_{21}}]\,[\underline{x_1}, \bar{x}_1] + [\underline{a_{22}}, \overline{a_{22}}]\,[\underline{x_2}, \bar{x}_2] + \ldots\ldots + [\underline{a_{2n}}, \overline{a_{2n}}]\,[\underline{x_n}, \bar{x}_n] = [\underline{b_2}, \bar{b}_2]$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$[\underline{a_{n1}}, \overline{a_{n1}}]\,[\underline{x_1}, \bar{x}_1] + [\underline{a_{n2}}, \overline{a_{n2}}]\,[\underline{x_2}, \bar{x}_2] + \ldots\ldots + [\underline{a_{nn}}, \overline{a_{nn}}]\,[\underline{x_n}, \bar{x}_n] = [\underline{b_n}, \bar{b}_n]$$

This $n \times n$ system can be extended into $2n \times 2n$ the system of linear equations as follows,

$$\begin{aligned} \underline{a_{11}}\,\underline{x_1} + \underline{a_{12}}\,\underline{x_2} + \ldots\ldots + \underline{a_{1n}}\,\underline{x_n} &= \underline{b_1} \\ \overline{a_{11}}\,\bar{x}_1 + \overline{a_{12}}\,\bar{x}_2 + \ldots\ldots + \overline{a_{1n}}\,\bar{x}_n &= \bar{b}_1 \\ \underline{a_{21}}\,\underline{x_1} + \underline{a_{22}}\,\underline{x_2} + \ldots\ldots + \underline{a_{2n}}\,\underline{x_n} &= \underline{b_2} \\ \overline{a_{21}}\,\bar{x}_1 + \overline{a_{22}}\,\bar{x}_2 + \ldots\ldots + \overline{a_{2n}}\,\bar{x}_n &= \bar{b}_2 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ \underline{a_{n1}}\,\underline{x_1} + \underline{a_{n2}}\,\underline{x_2} + \ldots\ldots + \underline{a_{nn}}\,\underline{x_n} &= \underline{b_n} \\ \overline{a_{n1}}\,\bar{x}_1 + \overline{a_{n2}}\,\bar{x}_2 + \ldots\ldots + \overline{a_{nn}}\,\bar{x}_n &= \bar{b}_n \end{aligned} \quad \ldots\ldots\ldots\ldots(1.4)$$

System (1.4) can be written as matrix form,

$$\begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} & \cdots\cdots & \underline{a_{1n}} & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & \cdots\cdots & 0 & \overline{a_{11}} & \overline{a_{12}} & \cdots\cdots & \overline{a_{1n}} \\ \cdots & \cdots & \cdots\cdots & \cdots & \cdots & \cdots & \cdots\cdots & \cdots \\ \underline{a_{n1}} & \underline{a_{n2}} & \cdots\cdots & \underline{a_{nn}} & 0 & 0 & \cdots\cdots & 0 \\ 0 & 0 & \cdots\cdots & 0 & \overline{a_{n1}} & \overline{a_{n2}} & \cdots\cdots & \overline{a_{nn}} \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ \cdot \\ \cdot \\ \underline{x_n} \\ \bar{x}_1 \\ \bar{x}_2 \\ \cdot \\ \cdot \\ \bar{x}_n \end{pmatrix} = \begin{pmatrix} \underline{b_1} \\ \bar{b}_1 \\ \underline{b_2} \\ \bar{b}_2 \\ \cdot \\ \cdot \\ \cdot \\ \underline{b_n} \\ \bar{b}_n \end{pmatrix}$$

For simplicity, we consider system (1.3) for two equations with two unknowns. Then we have,

$$\begin{pmatrix} \underline{a_{11}} & \underline{a_{12}} & 0 & 0 \\ 0 & 0 & \overline{a_{11}} & \overline{a_{12}} \\ \underline{a_{21}} & \underline{a_{22}} & 0 & 0 \\ 0 & 0 & \overline{a_{21}} & \overline{a_{22}} \end{pmatrix} \begin{pmatrix} \underline{x_1} \\ \underline{x_2} \\ \bar{x}_1 \\ \bar{x}_2 \end{pmatrix} = \begin{pmatrix} \underline{b_1} \\ \bar{b}_1 \\ \underline{b_2} \\ \bar{b}_2 \end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots(1.5)$$

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From system (1.5) we get,

$$\begin{aligned}\underline{a_{11}}\,\underline{x_1} + \underline{a_{12}}\,\underline{x_2} &= \underline{b_1},\\ \overline{a_{11}}\,\overline{x_1} + \overline{a_{12}}\,\overline{x_2} &= \overline{b_1},\\ \underline{a_{21}}\,\underline{x_1} + \underline{a_{22}}\,\underline{x_2} &= \underline{b_2},\\ \overline{a_{21}}\,\overline{x_1} + \overline{a_{22}}\,\overline{x_2} &= \overline{b_2},\end{aligned} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.6)$$

First and second equations of the system (1.6) can be written in matrix form as,

$$\begin{pmatrix}\underline{a_{11}} & \underline{a_{12}}\\ \underline{a_{21}} & \underline{a_{22}}\end{pmatrix}\begin{pmatrix}\underline{x_1}\\ \underline{x_2}\end{pmatrix} = \begin{pmatrix}\underline{b_1}\\ \underline{b_2}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\underline{x_1}\\ \underline{x_2}\end{pmatrix} = \begin{pmatrix}\underline{a_{11}} & \underline{a_{12}}\\ \underline{a_{21}} & \underline{a_{22}}\end{pmatrix}^{-1}\begin{pmatrix}\underline{b_1}\\ \underline{b_2}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\underline{x_1}\\ \underline{x_2}\end{pmatrix} = \frac{1}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}\begin{pmatrix}\underline{a_{22}} & -\underline{a_{12}}\\ -\underline{a_{21}} & \underline{a_{11}}\end{pmatrix}\begin{pmatrix}\underline{b_1}\\ \underline{b_2}\end{pmatrix}$$

$$\Rightarrow \begin{pmatrix}\underline{x_1}\\ \underline{x_2}\end{pmatrix} = \begin{pmatrix}\dfrac{\underline{a_{22}}\,\underline{b_1} - \underline{a_{12}}\,\underline{b_2}}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}\\[2ex] \dfrac{\underline{a_{11}}\,\underline{b_2} - \underline{a_{21}}\,\underline{b_1}}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}\end{pmatrix}$$

$$\therefore \underline{x_1} = \frac{\underline{a_{22}}\,\underline{b_1} - \underline{a_{12}}\,\underline{b_2}}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}$$

$$\underline{x_2} = \frac{\underline{a_{11}}\,\underline{b_2} - \underline{a_{21}}\,\underline{b_1}}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}.$$

It should be noted that,

$$\left[(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}}) \neq 0\right].$$

Similarly, from equations first and a second system (1.6), we have,

$$\overline{x_1} = \frac{\overline{a_{22}}\,\overline{b_1} - \overline{a_{12}}\,\overline{b_2}}{(\overline{a_{11}}\;\overline{a_{22}} - \overline{a_{12}}\;\overline{a_{21}})}$$

$$\overline{x_2} = \frac{\overline{a_{11}}\,\overline{b_2} - \overline{a_{21}}\,\overline{b_1}}{(\overline{a_{11}}\;\overline{a_{22}} - \overline{a_{12}}\;\overline{a_{21}})}.$$

Hence the solution of system (1.2) can be written in the fuzzy interval as,

$$[\underline{x_1}, \overline{x_1}] = \left[\frac{\underline{a_{22}}\,\underline{b_1} - \underline{a_{12}}\,\underline{b_2}}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}, \frac{\overline{a_{22}}\,\overline{b_1} - \overline{a_{12}}\,\overline{b_2}}{(\overline{a_{11}}\;\overline{a_{22}} - \overline{a_{12}}\;\overline{a_{21}})}\right]$$

$$[\underline{x_2}, \overline{x_2}] = \left[\frac{\underline{a_{11}}\,\underline{b_2} - \underline{a_{21}}\,\underline{b_1}}{(\underline{a_{11}}\;\underline{a_{22}} - \underline{a_{21}}\;\underline{a_{12}})}, \frac{\overline{a_{11}}\,\overline{b_2} - \overline{a_{21}}\,\overline{b_1}}{(\overline{a_{11}}\;\overline{a_{22}} - \overline{a_{12}}\;\overline{a_{21}})}\right].$$

Thus we get the fuzzy solution for a 2×2 fully fuzzy system of linear equations. Similarly, we can find an unknown fuzzy vector for a $n\times n$ fully fuzzy system of linear equations.

4 Applications

4.1 Solving fuzzy system using fuzzy matrix equation system [6][7][8]

Consider a 3×3 fuzzy matrix of the system as follows,

$$\begin{pmatrix}2 & 4 & -6\\ 1 & 0 & -1\\ 3 & 5 & 7\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}(1/2/4)\\ (-2/1/3)\\ (6/8/12)\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.2)$$

We solve this fuzzy system using fuzzy matrix equation system.

To solve (1.2), we need to convert each triangular fuzzy number to fuzzy interval using α-cut.

$$\begin{aligned}(1/2/4) &= [(2-1)\alpha + 1, (2-4)\alpha + 4] = [1+\alpha, 4-2\alpha],\\ (-2/1/3) &= [(1-(-2))\alpha + (-2), (1-3)\alpha + 3] = [-2+3\alpha, 3-2\alpha],\\ (6/8/12) &= [(8-6)\alpha + 6, (8-12)\alpha + 12] = [6+2\alpha, 12-4\alpha].\end{aligned}$$

Now (1.2) becomes,

$$\begin{pmatrix}2 & 4 & -6\\ 1 & 0 & -1\\ 3 & 5 & 7\end{pmatrix}\begin{pmatrix}x\\ y\\ z\end{pmatrix} = \begin{pmatrix}[1+\alpha, 4-2\alpha]\\ [-2+3\alpha, 3-2\alpha]\\ [6+2\alpha, 12-4\alpha]\end{pmatrix} \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(1.3)$$

Comparing equation (1.3) with the general form of fuzzy system $AX = B$ We get ,

$\begin{pmatrix}2 & 4 & -6\\ 1 & 0 & -1\\ 3 & 5 & 7\end{pmatrix}$ which is a 3×3 crisp matrix.

$\begin{pmatrix}[1+\alpha, 4-2\alpha]\\ [-2+3\alpha, 3-2\alpha]\\ [6+2\alpha, 12-4\alpha]\end{pmatrix}$ is a 3×1 fuzzy matrix and $X = \begin{pmatrix}x\\ y\\ z\end{pmatrix}$ is an 3×1 unknown fuzzy matrix. To construct the matrix S from the system (1.3) we have,

$\begin{pmatrix}2 & 4 & 0 & 0 & 0 & -6\\ 1 & 0 & 0 & 0 & 0 & -1\\ 3 & 5 & 7 & 0 & 0 & 0\\ 0 & 0 & -6 & 2 & 4 & 0\\ 0 & 0 & -1 & 1 & 0 & 0\\ 0 & 0 & 0 & 3 & 5 & 7\end{pmatrix}$ which is a 6×6 crisp matrix, and

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$$X=\begin{pmatrix}\underline{x}\\ \underline{y}\\ \underline{z}\\ \overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix}, Y=\begin{pmatrix}1+\alpha\\ -2+3\alpha\\ 6+2\alpha\\ 4-2\alpha\\ 3-2\alpha\\ 12-4\alpha\end{pmatrix}$$ .Here, $S_1=\begin{pmatrix}2&4&0\\1&0&0\\3&5&7\end{pmatrix}$,and

$S_2=\begin{pmatrix}0&0&-4\\0&0&-1\\0&0&0\end{pmatrix}$ So, $S=\begin{pmatrix}S_1&S_2\\S_2&S_1\end{pmatrix}$ and

$$S^{-1}=\begin{pmatrix}-\frac{2}{3}&\frac{11}{15}&\frac{8}{15}&\frac{7}{12}&\frac{7}{30}&-\frac{7}{15}\\ -\frac{5}{12}&-\frac{23}{30}&\frac{8}{15}&\frac{7}{12}&\frac{7}{30}&-\frac{7}{15}\\ \frac{7}{12}&\frac{7}{30}&-\frac{7}{15}&-\frac{2}{3}&-\frac{4}{15}&\frac{8}{15}\\ \frac{7}{12}&\frac{7}{30}&-\frac{7}{15}&-\frac{2}{3}&\frac{11}{15}&\frac{8}{15}\\ \frac{7}{12}&\frac{7}{30}&-\frac{7}{15}&-\frac{5}{12}&-\frac{23}{30}&\frac{8}{15}\\ -\frac{2}{3}&-\frac{4}{15}&\frac{8}{15}&\frac{7}{12}&\frac{7}{30}&-\frac{7}{15}\end{pmatrix}$$

We use **Mathematica 5.0** to find the inverse.

Now for the system $SX=Y$ we have,

$$\begin{pmatrix}2&4&0&0&0&-6\\1&0&0&0&0&-1\\3&5&7&0&0&0\\0&0&-6&2&4&0\\0&0&-1&1&0&0\\0&0&0&3&5&7\end{pmatrix}\begin{pmatrix}\underline{x}\\ \underline{y}\\ \underline{z}\\ \overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix}=\begin{pmatrix}1+\alpha\\ -2+3\alpha\\ 6+2\alpha\\ 4-2\alpha\\ 3-2\alpha\\ 12-4\alpha\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\underline{x}\\ \underline{y}\\ \underline{z}\\ \overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix}=\begin{pmatrix}-\frac{2}{3}&\frac{11}{15}&\frac{8}{15}&\frac{7}{12}&\frac{7}{30}&-\frac{7}{15}\\ -\frac{5}{12}&-\frac{23}{30}&\frac{8}{15}&\frac{7}{12}&\frac{7}{30}&-\frac{7}{15}\\ \frac{7}{12}&\frac{7}{30}&-\frac{7}{15}&-\frac{2}{3}&-\frac{4}{15}&\frac{8}{15}\\ \frac{7}{12}&\frac{7}{30}&-\frac{7}{15}&-\frac{2}{3}&\frac{11}{15}&\frac{8}{15}\\ \frac{7}{12}&\frac{7}{30}&-\frac{7}{15}&-\frac{5}{12}&-\frac{23}{30}&\frac{8}{15}\\ -\frac{2}{3}&-\frac{4}{15}&\frac{8}{15}&\frac{7}{12}&\frac{7}{30}&-\frac{7}{15}\end{pmatrix}\begin{pmatrix}1+\alpha\\ -2+3\alpha\\ 6+2\alpha\\ 4-2\alpha\\ 3-2\alpha\\ 12-4\alpha\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\underline{x}\\ \underline{y}\\ \underline{z}\\ \overline{x}\\ \overline{y}\\ \overline{z}\end{pmatrix}=\begin{pmatrix}\frac{(-9+17\alpha)}{6}\\ \frac{(21-17\alpha)}{12}\\ \frac{(3+\alpha)}{12}\\ \frac{(39-23\alpha)}{12}\\ \frac{(-3+7\alpha)}{12}\\ \frac{(3-\alpha)}{6}\end{pmatrix}$$

Hence the solution of the system (1.2) is

$$x=\left(\frac{(-9+17\alpha)}{6},\frac{(39-23\alpha)}{12}\right)$$
$$y=\left(\frac{(21-17\alpha)}{12},\frac{(-3+7\alpha)}{12}\right)$$
$$z=\left(\frac{(3+\alpha)}{12},\frac{(3-\alpha)}{6}\right)$$

4.2 Solving triangular matrix equation using fuzzy matrix equation system [10]

$$\begin{pmatrix}(0.2,0.5,0.8)&(2,2.4,2.9)&(0.11,0.3,0.9)\\(0.12,0.15,0.2)&(0.09,0.15,0.22)&(6,6.1,6.2)\\(5,5.1,5.4)&(0.2,0.3,0.5)&(0.12,0.2,0.34)\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}(0.01,0.1,0.2)\\(0.11,0.16,0.2)\\(0.1,0.16,0.2)\end{pmatrix}.$$

Which can be written as follows,

$$\begin{aligned}(0.2,0.5,0.8)x_1+(2,2.4,2.9)x_2+(0.11,0.3,0.9)x_3&=(0.01,0.1,0.2)\\(0.12,0.15,0.2)x_1+(0.09,0.15,0.22)x_2+(6,6.1,6.2)x_3&=(0.11,0.16,0.2)\\(5,5.1,5.4)x_1+(0.2,0.3,0.5)x_2+(0.12,0.2,0.34)x_3&=(0.1,0.16,0.2)\end{aligned}$$ ……………..….(3.1)

Using the α-cuts of the unknowns the above system can be written as,

$$\begin{aligned}(0.2,0.5,0.8)[\underline{x}_1,\bar{x}_1]+(2,2.4,2.9)[\underline{x}_2,\bar{x}_2]+(0.11,0.3,0.9)[\underline{x}_3,\bar{x}_3]&=(0.01,0.1,0.2)\\(0.12,0.15,0.2)[\underline{x}_1,\bar{x}_1]+(0.09,0.15,0.22)[\underline{x}_2,\bar{x}_2]+(6,6.1,6.2)[\underline{x}_3,\bar{x}_3]&=(0.11,0.16,0.2)\\(5,5.1,5.4)[\underline{x}_1,\bar{x}_1]+(0.2,0.3,0.5)[\underline{x}_2,\bar{x}_2]+(0.12,0.2,0.34)[\underline{x}_3,\bar{x}_3]&=(0.1,0.16,0.2)\end{aligned}$$

Now convert each triangular fuzzy number into fuzzy interval using α-cut,

$$(0.2,0.5,0.8)=[(0.5-0.2)\alpha+0.2,(0.5-0.8)\alpha+0.8]=[0.2+0.3\alpha,0.8-0.3\alpha]$$

$$(2,2.4,2.9)=[(2.4-2)\alpha+2,(2.4-2.9)\alpha+2.9]=[2+0.4\alpha,2.9-0.5\alpha]$$

$$(0.11,0.3,0.9)=[(0.3-0.11)\alpha+0.11,(0.3-0.9)\alpha+0.9]=[0.11+0.19\alpha,0.9-0.6\alpha]$$

$$(0.01,0.1,0.2)=[(0.1-0.01)\alpha+0.01,(0.1-0.2)\alpha+0.2]=[0.01+0.09\alpha,0.2-0.1\alpha]$$

$$(0.12,0.15,0.2)=[(0.15-0.12)\alpha+0.12,(0.15-0.2)\alpha+0.2]=[0.12+0.03\alpha,0.2-0.05\alpha]$$

$$(0.09,0.15,0.22)=[(0.15-0.09)\alpha+0.09,(0.15-0.22)\alpha+0.22]=[0.09+0.06\alpha,0.22-0.07\alpha]$$

$$(6,6.1,6.2)=[(6.1-6)\alpha+6,(6.1-6.2)\alpha+6.2]=[6+0.1\alpha,6.2-0.1\alpha]$$

$$(0.11,0.16,0.2)=[(0.16-0.11)\alpha+0.11,(0.16-0.2)\alpha+0.2]=[0.11+0.05\alpha,0.2-0.04\alpha]$$

$$(5,5.1,5.4)=[(5.1-5)\alpha+5,(5.1-5.4)\alpha+5.4]=[5+0.1\alpha,5.4-0.3\alpha]$$

$$(0.2,0.3,0.5)=[(0.3-0.2)\alpha+0.2,(0.3-0.5)\alpha+0.5]=[0.2+0.1\alpha,0.5-0.2\alpha]$$

$$(0.12,0.2,0.34)=[(0.2-0.12)\alpha+0.12,(0.2-0.34)\alpha+0.34]=[0.12+0.8\alpha,0.34-0.14\alpha]$$

$$(0.1,0.16,0.2)=[(0.16-0.1)\alpha+0.1,(0.16-0.2)\alpha+0.2]=[0.1+0.06\alpha,0.2-0.04\alpha]$$

Now we have converted all triangular fuzzy number into the fuzzy interval.
The system (3.1) implies,

$$[0.2+0.3\alpha,0.8-0.3\alpha][\underline{x}_1,\bar{x}_1]+[2+0.4\alpha,2.9-0.5\alpha][\underline{x}_2,\bar{x}_2]+[0.11+0.19\alpha,\ 0.9-0.6\alpha][\underline{x}_3,\bar{x}_3]=[0.01+0.09\alpha,0.2-0.1\alpha]$$

$$[0.12+0.03\alpha,0.2-0.05\alpha][\underline{x}_1,\bar{x}_1]+[0.09+0.06\alpha,0.22-0.07\alpha][\underline{x}_2,\bar{x}_2]+[6+0.1\alpha,6.2-0.1\alpha][\underline{x}_3,\bar{x}_3]=[0.11+0.05\alpha,0.2-0.04\alpha]$$

$$[5+0.1\alpha,5.4-0.3\alpha][\underline{x}_1,\bar{x}_1]+[0.2+0.1\alpha,0.5-0.2\alpha][\underline{x}_2,\bar{x}_2]+[0.12+0.8\alpha,\ 0.34-0.14\alpha][\underline{x}_3,\bar{x}_3]=[0.1+0.06\alpha,0.2-0.04\alpha]$$

Or,

$$\begin{bmatrix}(0.2+0.3\alpha)\underline{x}_1+(2+0.4\alpha)\underline{x}_2+(0.11+0.19\alpha)\underline{x}_3,\\ (0.8-0.3\alpha)\bar{x}_1+(2.9-0.5\alpha)\bar{x}_2+(0.9-0.6\alpha)\bar{x}_3\end{bmatrix}=[0.01+0.09\alpha,0.2-0.1\alpha]$$

$$\begin{bmatrix}(0.12+0.03\alpha)\underline{x}_1+(0.09+0.06\alpha)\underline{x}_2+(6+0.1\alpha)\underline{x}_3,\\ (0.2-0.05\alpha)\bar{x}_1+(0.22-0.07\alpha)\bar{x}_2+(6.2-0.1\alpha)\bar{x}_3\end{bmatrix}=[0.11+0.05\alpha,0.2-0.04\alpha]$$

$$[(5+0.1\alpha)\underline{x}_1+(0.2+0.1\alpha)\underline{x}_2+(0.12+0.8\alpha)\underline{x}_3,(5.4-0.3\alpha)\bar{x}_1+(0.5-0.2\alpha)\bar{x}_2+(0.34-0.14\alpha)\bar{x}_3]=[0.1+0.06\alpha,0.2-0.04\alpha]$$

The above system can be partitioned into two systems,

$$\begin{aligned}(0.2+0.3\alpha)\underline{x}_1+(2+0.4\alpha)\underline{x}_2+(0.11+0.19\alpha)\underline{x}_3&=0.01+0.09\alpha\\(0.12+0.03\alpha)\underline{x}_1+(0.09+0.06\alpha)\underline{x}_2+(6+0.1\alpha)\underline{x}_3&=0.11+0.05\alpha\\(5+0.1\alpha)\underline{x}_1+(0.2+0.1\alpha)\underline{x}_2+(0.12+0.8\alpha)\underline{x}_3&=0.1+0.06\alpha\end{aligned} \quad \text{……..(3.2)}$$

And

$$\begin{aligned}(0.8-0.3\alpha)\bar{x}_1+(2.9-0.5\alpha)\bar{x}_2+(0.9-0.6\alpha)\bar{x}_3&=0.2-0.1\alpha\\(0.2-0.05\alpha)\bar{x}_1+(0.22-0.07\alpha)\bar{x}_2+(6.2-0.1\alpha)\bar{x}_3&=0.2-0.04\alpha\\(5.4-0.3\alpha)\bar{x}_1+(0.5-0.2\alpha)\bar{x}_2+(0.34-0.14\alpha)\bar{x}_3&=0.2-0.04\alpha\end{aligned} \quad \text{……..(3.3)}$$

In matrix form (3.2) can be written as,

$$\begin{pmatrix}(0.2+0.3\alpha)&(2+0.4\alpha)&(0.11+0.19\alpha)\\(0.12+0.03\alpha)&(0.09+0.06\alpha)&(6+0.1\alpha)\\(5+0.1\alpha)&(0.2+0.1\alpha)&(0.12+0.8\alpha)\end{pmatrix}\begin{pmatrix}\underline{x}_1\\\underline{x}_2\\\underline{x}_3\end{pmatrix}=\begin{pmatrix}0.01+0.09\alpha\\0.11+0.05\alpha\\0.1+0.06\alpha\end{pmatrix}$$

Now we solve the above system using matrix inversion

$$\begin{pmatrix}\underline{x}_1\\\underline{x}_2\\\underline{x}_3\end{pmatrix}=\begin{pmatrix}(0.2+0.3\alpha)&(2+0.4\alpha)&(0.11+0.19\alpha)\\(0.12+0.03\alpha)&(0.09+0.06\alpha)&(6+0.1\alpha)\\(5+0.1\alpha)&(0.2+0.1\alpha)&(0.12+0.8\alpha)\end{pmatrix}^{-1}\begin{pmatrix}0.01+0.09\alpha\\0.11+0.05\alpha\\0.1+0.06\alpha\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\underline{x}_1\\\underline{x}_2\\\underline{x}_3\end{pmatrix}=\begin{pmatrix}\dfrac{1.163138+0.677818\alpha-0.003914\alpha^2-0.009914\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3}\\\dfrac{0.123316+2.34766\alpha-0.034108\alpha^2+0.008332\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3}\\\dfrac{1.06914+0.66973\alpha+0.07734\alpha^2+0.00059\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3}\end{pmatrix} \quad \text{……….(3.4)}$$

Also, Equation (3.3) can be written as,

$$\begin{pmatrix}(0.8-0.3\alpha)&(2.9-0.5\alpha)&(0.9-0.6\alpha)\\(0.2-0.05\alpha)&(0.22-0.07\alpha)&(6.2-0.1\alpha)\\(5.4-0.3\alpha)&(0.5-0.2\alpha)&(0.34-0.14\alpha)\end{pmatrix}\begin{pmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{pmatrix}=\begin{pmatrix}0.2-0.1\alpha\\0.2-0.04\alpha\\0.2-0.04\alpha\end{pmatrix}$$

Now we solve the above system using matrix inversion

$$\begin{pmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{pmatrix}=\begin{pmatrix}(0.8-0.3\alpha)&(2.9-0.5\alpha)&(0.9-0.6\alpha)\\(0.2-0.05\alpha)&(0.22-0.07\alpha)&(6.2-0.1\alpha)\\(5.4-0.3\alpha)&(0.5-0.2\alpha)&(0.34-0.14\alpha)\end{pmatrix}^{-1}\begin{pmatrix}0.2-0.1\alpha\\0.2-0.04\alpha\\0.2-0.04\alpha\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{pmatrix}=\begin{pmatrix}\dfrac{2.84416-0.76004\alpha+0.00998\alpha^2-0.0013\alpha^3}{93.4954-20.6457\alpha+0.62812\alpha^2-0.00184\alpha^3}\\\dfrac{4.8088-2.42328\alpha+0.02966\alpha^2-0.00078\alpha^3}{93.4954-20.6457\alpha+0.62812\alpha^2-0.00184\alpha^3}\\\dfrac{2.7536-1.03704\alpha+0.10258\alpha^2-0.00234\alpha^3}{93.4954-20.6457\alpha+0.62812\alpha^2-0.00184\alpha^3}\end{pmatrix} \quad \text{………(3.5)}$$

Hence the fuzzy solution of the system (3.1) is

$$[\underline{x}_1,\bar{x}_1]=\left[\frac{1.163138+0.677818\alpha-0.003914\alpha^2-0.009914\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3},\frac{2.84416-0.76004\alpha+0.00998\alpha^2-0.0013\alpha^3}{93.4954-20.6457\alpha+0.62812\alpha^2-0.00184\alpha^3}\right]$$

$$[\underline{x}_2,\bar{x}_2]=\left[\frac{0.123316+2.34766\alpha-0.034108\alpha^2+0.008332\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3},\frac{4.8088-2.42328\alpha+0.02966\alpha^2-0.00078\alpha^3}{93.4954-20.6457\alpha+0.62812\alpha^2-0.00184\alpha^3}\right]$$

$$[\underline{x}_3,\bar{x}_3]=\left[\frac{1.06914+0.66973\alpha+0.07734\alpha^2+0.00059\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3},\frac{2.7536-1.03704\alpha+0.10258\alpha^2-0.00234\alpha^3}{93.4954-20.6457\alpha+0.62812\alpha^2-0.00184\alpha^3}\right]$$

4.3 Solving Trapezoidal matrix equation using fuzzy matrix equation system

Consider the following trapezoidal matrix equation system with three unknowns

$$\begin{pmatrix}(0.2,0.5,0.6,0.8)&(2,2.4,2.6,2.9)&(0.11,0.3,0.5,0.9)\\(0.12,0.15,0.18,0.2)&(0.09,0.15,0.17,0.22)&(6,6.1,6.14,6.2)\\(5,5.1,5.2,5.4)&(0.2,0.3,0.4,0.5)&(0.12,0.2,0.25,0.34)\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}(0.01,0.1,0.15,0.2)\\(0.11,0.16,0.18,0.2)\\(0.1,0.16,0.18,0.2)\end{pmatrix} \quad \text{……(3.6)}$$

Which can be written as follows,

$$\begin{aligned}(0.2,0.5,0.6,0.8)x_1+(2,2.4,2.6,2.9)x_2+(0.11,0.3,0.5,0.9)x_3&=(0.01,0.1,0.15,0.2)\\(0.12,0.15,0.18,0.2)x_1+(0.09,0.15,0.17,0.22)x_2+(6,6.1,6.14,6.2)x_3&=(0.11,0.16,0.18,0.2)\\(5,5.1,5.2,5.4)x_1+(0.2,0.3,0.4,0.5)x_2+(0.12,0.2,0.25,0.34)x_3&=(0.1,0.16,0.18,0.2)\end{aligned}$$

Using the $\alpha-$cut of x_1, x_2 and x_3 we get,

$$\begin{aligned}(0.2,0.5,0.6,0.8)[\underline{x}_1,\bar{x}_1]+(2,2.4,2.6,2.9)[\underline{x}_2,\bar{x}_2]+(0.11,0.3,0.5,0.9)[\underline{x}_3,\bar{x}_3]&=(0.01,0.1,0.15,0.2)\\(0.12,0.15,0.18,0.2)[\underline{x}_1,\bar{x}_1]+(0.09,0.15,0.17,0.22)[\underline{x}_2,\bar{x}_2]+(6,6.1,6.14,6.2)[\underline{x}_3,\bar{x}_3]&=(0.11,0.16,0.18,0.2)\\(5,5.1,5.2,5.4)[\underline{x}_1,\bar{x}_1]+(0.2,0.3,0.4,0.5)[\underline{x}_2,\bar{x}_2]+(0.12,0.2,0.25,0.34)[\underline{x}_3,\bar{x}_3]&=(0.1,0.16,0.18,0.2)\end{aligned}$$

Now covert each triangular fuzzy number into fuzzy interval using α-cut,

$$(0.2,0.5,0.6,0.8)=[(0.5-0.2)\alpha+0.2,(0.6-0.8)\alpha+0.8]=[0.2+0.3\alpha,0.8-0.2\alpha]$$

$$(2,2.4,2.6,2.9)=[(2.4-2)\alpha+2,(2.6-2.9)\alpha+2.9]=[2+0.4\alpha,2.9-0.3\alpha]$$

$$(0.11,0.3,0.5,0.9)=[(0.3-0.11)\alpha+0.11,(0.5-0.9)\alpha+0.9]=[0.11+0.19\alpha,0.9-0.4\alpha]$$

$$(0.01,0.1,0.15,0.2)=[(0.1-0.01)\alpha+0.01,(0.15-0.2)\alpha+0.2]=[0.01+0.09\alpha,0.2-0.05\alpha]$$

$$(0.12,0.15,0.18,0.2)=[(0.15-0.12)\alpha+0.12,(0.18-0.2)\alpha+0.2]=[0.12+0.03\alpha,0.2-0.02\alpha]$$

$$(0.09,0.15,0.17,0.22)=[(0.15-0.09)\alpha+0.09,(0.17-0.22)\alpha+0.22]=[0.09+.06\alpha,0.22-.05\alpha]$$

$$(6,6.1,6.14,6.2)=[(6.1-6)\alpha+6,(6.14-6.2)\alpha+6.2]=[6+0.1\alpha,6.2-0.06\alpha]$$

$$(0.11,0.16,0.18,0.2)=[(0.16-0.11)\alpha+0.11,(0.18-0.2)\alpha+0.2]=[0.11+0.05\alpha,0.2-0.02\alpha]$$

$$(5,5.1,5.2,5.4)=[(5.1-5)\alpha+5,(5.2-5.4)\alpha+5.4]=[5+0.1\alpha,5.4-0.2\alpha]$$

$$(0.2,0.3,0.4,0.5)=[(0.3-0.2)\alpha+0.2,(0.4-0.5)\alpha+0.5]=[0.2+0.1\alpha,0.5-0.1\alpha]$$

$$(0.12,0.2,0.25,0.34)=[(0.2-0.12)\alpha+0.12,(0.25-0.34)\alpha+0.34]=[0.12+0.8\alpha,0.34-0.09\alpha]$$

$$(0.1,0.16,0.18,0.2)=[(0.16-0.1)\alpha+0.1,(0.18-0.2)\alpha+0.2]=[0.1+0.06\alpha,0.2-0.02\alpha]$$

Now we have converted all triangular fuzzy number into the fuzzy interval.
The system (3.2) implies,

$$[0.2+0.3\alpha, 0.8-0.2\alpha][\underline{x}_1,\bar{x}_1]+[2+0.4\alpha, 2.9-0.3\alpha][\underline{x}_2,\bar{x}_2]+[0.11+0.19\alpha, 0.9-0.4\alpha][\underline{x}_3,\bar{x}_3]=[0.01+0.09\alpha, 0.2-0.05\alpha]$$

$$[0.12+0.03\alpha, 0.2-0.02\alpha][\underline{x}_1,\bar{x}_1]+[0.09+0.06\alpha, 0.22-0.05\alpha][\underline{x}_2,\bar{x}_2]+[6+0.1\alpha, 6.2-0.06\alpha][\underline{x}_3,\bar{x}_3]=[0.11+0.05\alpha, 0.2-0.02\alpha]$$

$$[5+0.1\alpha, 5.4-0.2\alpha][\underline{x}_1,\bar{x}_1]+[0.2+0.1\alpha, 0.5-0.1\alpha][\underline{x}_2,\bar{x}_2]+[0.12+0.8\alpha, 0.34-0.09\alpha][\underline{x}_3,\bar{x}_3]=[0.1+0.06\alpha, 0.2-0.02\alpha]$$

Or

$$\begin{bmatrix}(0.2+0.3\alpha)\underline{x}_1+(2+0.4\alpha)\underline{x}_2+(0.11+0.19\alpha)\underline{x}_3,\\ (0.8-0.2\alpha)\bar{x}_1+(2.9-0.3\alpha)\bar{x}_2+(0.9-0.4\alpha)\bar{x}_3\end{bmatrix}=[0.01+0.09\alpha, 0.2-0.05\alpha]$$

$$\begin{bmatrix}(0.12+0.03\alpha)\underline{x}_1+(0.09+0.06\alpha)\underline{x}_2+(6+0.1\alpha)\underline{x}_3,\\ (0.2-0.02\alpha)\bar{x}_1+(0.22-0.05\alpha)\bar{x}_2+(6.2-0.06\alpha)\bar{x}_3\end{bmatrix}=[0.11+0.05\alpha, 0.2-0.02\alpha]$$

$$[(5+0.1\alpha)\underline{x}_1+(0.2+0.1\alpha)\underline{x}_2+(0.12+0.8\alpha)\underline{x}_3, (5.4-0.2\alpha)\bar{x}_1+(0.5-0.1\alpha)\bar{x}_2+(0.34-0.09\alpha)\bar{x}_3]=[0.1+0.06\alpha, 0.2-0.02\alpha]$$

The above system can be partitioned into two systems,

$$\begin{aligned}(0.2+0.3\alpha)\underline{x}_1+(2+0.4\alpha)\underline{x}_2+(0.11+0.19\alpha)\underline{x}_3&=0.01+0.09\alpha\\(0.12+0.03\alpha)\underline{x}_1+(0.09+0.06\alpha)\underline{x}_2+(6+0.1\alpha)\underline{x}_3&=0.11+0.05\alpha\\(5+0.1\alpha)\underline{x}_1+(0.2+0.1\alpha)\underline{x}_2+(0.12+0.8\alpha)\underline{x}_3&=0.1+0.06\alpha\end{aligned}$$ ……....(3.7)

$$\begin{aligned}(0.8-0.2\alpha)\bar{x}_1+(2.9-0.3\alpha)\bar{x}_2+(0.9-0.4\alpha)\bar{x}_3&=0.2-0.05\alpha\\(0.2-0.02\alpha)\bar{x}_1+(0.22-0.05\alpha)\bar{x}_2+(6.2-0.06\alpha)\bar{x}_3&=0.2-0.02\alpha\\(5.4-0.2\alpha)\bar{x}_1+(0.5-0.1\alpha)\bar{x}_2+(0.34-0.09\alpha)\bar{x}_3&=0.2-0.02\alpha\end{aligned}$$ ………(3.8)

In matrix form (3.7) can be written as,

$$\begin{pmatrix}(0.2+0.3\alpha)&(2+0.4\alpha)&(0.11+0.19\alpha)\\(0.12+0.03\alpha)&(0.09+0.06\alpha)&(6+0.1\alpha)\\(5+0.1\alpha)&(0.2+0.1\alpha)&(0.12+0.8\alpha)\end{pmatrix}\begin{pmatrix}\underline{x}_1\\\underline{x}_2\\\underline{x}_3\end{pmatrix}=\begin{pmatrix}0.01+0.09\alpha\\0.11+0.05\alpha\\0.1+0.06\alpha\end{pmatrix}$$

Now we solve the above system using matrix inversion

$$\begin{pmatrix}\underline{x}_1\\\underline{x}_2\\\underline{x}_3\end{pmatrix}=\begin{pmatrix}(0.2+0.3\alpha)&(2+0.4\alpha)&(0.11+0.19\alpha)\\(0.12+0.03\alpha)&(0.09+0.06\alpha)&(6+0.1\alpha)\\(5+0.1\alpha)&(0.2+0.1\alpha)&(0.12+0.8\alpha)\end{pmatrix}^{-1}\begin{pmatrix}0.01+0.09\alpha\\0.11+0.05\alpha\\0.1+0.06\alpha\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\underline{x}_1\\\underline{x}_2\\\underline{x}_3\end{pmatrix}=\begin{pmatrix}\dfrac{1.163138+0.677818\alpha-0.003914\alpha^2-0.009914\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3}\\\dfrac{0.123316+2.34766\alpha-0.034108\alpha^2+0.008332\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3}\\\dfrac{1.06914+0.66973\alpha+0.07734\alpha^2+0.00059\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3}\end{pmatrix}$$ ……..(3.9)

Also (3.8) can be written as,

$$\begin{pmatrix}(0.8-0.2\alpha)&(2.9-0.3\alpha)&(0.9-0.4\alpha)\\(0.2-0.02\alpha)&(0.22-0.05\alpha)&(6.2-0.06\alpha)\\(5.4-0.2\alpha)&(0.5-0.1\alpha)&(0.34-0.09\alpha)\end{pmatrix}\begin{pmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{pmatrix}=\begin{pmatrix}0.2-0.05\alpha\\0.2-0.02\alpha\\0.2-0.02\alpha\end{pmatrix}$$

Now we solve the above system using matrix inversion

$$\begin{pmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{pmatrix}=\begin{pmatrix}(0.8-0.2\alpha)&(2.9-0.3\alpha)&(0.9-0.4\alpha)\\(0.2-0.02\alpha)&(0.22-0.05\alpha)&(6.2-0.06\alpha)\\(5.4-0.2\alpha)&(0.5-0.1\alpha)&(0.34-0.09\alpha)\end{pmatrix}^{-1}\begin{pmatrix}0.2-0.05\alpha\\0.2-0.02\alpha\\0.2-0.02\alpha\end{pmatrix}$$

$$\Rightarrow\begin{pmatrix}\bar{x}_1\\\bar{x}_2\\\bar{x}_3\end{pmatrix}=\begin{pmatrix}\dfrac{2.84416-0.43662\alpha+0.0078\alpha^2-0.000145\alpha^3}{93.4954-12.7009\alpha+0.2467\alpha^2+0.00044\alpha^3}\\\dfrac{4.8088-1.11308\alpha-0.002\alpha^2+0.00081\alpha^3}{93.4954-12.7009\alpha+0.2467\alpha^2+0.00044\alpha^3}\\\dfrac{2.7536-0.58312\alpha+0.03272\alpha^2-0.00048\alpha^3}{93.4954-12.7009\alpha+0.2467\alpha^2+0.00044\alpha^3}\end{pmatrix}$$ ….....(3.10)

Here, we compute the inverse using **Mathematica 5.0**.

Hence the fuzzy solution of the system (3.6) is

$$[\underline{x}_1,\bar{x}_1]=\left[\frac{1.163138+0.677818\alpha-0.003914\alpha^2-0.009914\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3},\frac{2.84416-0.43662\alpha+0.0078\alpha^2-0.000145\alpha^3}{93.4954-12.7009\alpha+0.2467\alpha^2+0.00044\alpha^3}\right]$$

$$[\underline{x}_2,\bar{x}_2]=\left[\frac{0.123316+2.34766\alpha-0.034108\alpha^2+0.008332\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3},\frac{4.8088-1.11308\alpha-0.002\alpha^2+0.00081\alpha^3}{93.4954-12.7009\alpha+0.2467\alpha^2+0.00044\alpha^3}\right]$$

$$[\underline{x}_3,\bar{x}_3]=\left[\frac{1.06914+0.66973\alpha+0.07734\alpha^2+0.00059\alpha^3}{59.6865+13.4172\alpha+0.1619\alpha^2+0.00523\alpha^3},\frac{2.7536-0.58312\alpha+0.03272\alpha^2-0.00048\alpha^3}{93.4954-12.7009\alpha+0.2467\alpha^2+0.00044\alpha^3}\right]$$

4.4 Solving Interval matrix equation using fully fuzzy linear equations [10]

Consider the following system of fully fuzzy linear equations

$$\begin{pmatrix}[1,9]&[10,17]&[3,5]\\[2,5]&[-1,2]&[3,4]\\[1,4]&[2,4]&[3,6]\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}=\begin{pmatrix}[1,3]\\[-1,1]\\[0,1]\end{pmatrix}$$

Which can be written in the following form

$$\begin{aligned}[1,9]x_1+[10,17]x_2+[3,5]x_3&=[1,3]\\[2,5]x_1+[-1,2]x_2+[3,4]x_3&=[-1,1]\\[1,4]x_1+[2,4]x_2+[3,6]x_3&=[0,1]\end{aligned}$$ ……………………..…(3.11)

Using the $\alpha-$cut of x_1, x_2 and x_3 we get,

$$\begin{aligned}[1,9][\underline{x}_1,\bar{x}_1]+[10,17][\underline{x}_2,\bar{x}_2]+[3,5][\underline{x}_3,\bar{x}_3]&=[1,3]\\[2,5][\underline{x}_1,\bar{x}_1]+[-1,2][\underline{x}_2,\bar{x}_2]+[3,4][\underline{x}_3,\bar{x}_3]&=[-1,1]\\[1,4][\underline{x}_1,\bar{x}_1]+[2,4][\underline{x}_2,\bar{x}_2]+[3,6][\underline{x}_3,\bar{x}_3]&=[0,1]\end{aligned}$$

$$\begin{aligned}&[\underline{x}_1,9\bar{x}_1]+[10\underline{x}_2,17\bar{x}_2]+[3\underline{x}_3,5\bar{x}_3]=[1,3]\\\Rightarrow&[2\underline{x}_1,5\bar{x}_1]+[-\underline{x}_2,2\bar{x}_2]+[3\underline{x}_3,4\bar{x}_3]=[-1,1]\\&[\underline{x}_1,4\bar{x}_1]+[2\underline{x}_2,4\bar{x}_2]+[3\underline{x}_3,6\bar{x}_3]=[0,1]\end{aligned}$$

$$\begin{aligned}&[\underline{x}_1+10\underline{x}_2+3\underline{x}_3, 9\bar{x}_1+17\bar{x}_2+5\bar{x}_3]=[1,3]\\\Rightarrow&[2\underline{x}_1-\underline{x}_2+3\underline{x}_3, 5\bar{x}_1+2\bar{x}_2+4\bar{x}_3]=[-1,1]\\&[\underline{x}_1+2\underline{x}_2+3\underline{x}_3, 4\bar{x}_1+4\bar{x}_2+6\bar{x}_3]=[0,1]\end{aligned}$$ ………..(3.12)

System (3.12) a be partitioned into the following two systems,

$$\begin{aligned}\underline{x}_1+10\underline{x}_2+3\underline{x}_3&=1\\2\underline{x}_1-\underline{x}_2+3\underline{x}_3&=-1\\\underline{x}_1+2\underline{x}_2+3\underline{x}_3&=0\end{aligned}$$ ………………………………………...(3.13)

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And

$$\begin{aligned} 9\bar{x}_1 + 17\bar{x}_2 + 5\bar{x}_3 &= 3 \\ 5\bar{x}_1 + 2\bar{x}_2 + 4\bar{x}_3 &= 1 \\ 4\bar{x}_1 + 4\bar{x}_2 + 6\bar{x}_3 &= 1 \end{aligned} \qquad \text{...(3.14)}$$

System (3.13) can be written in matrix form as follows,

$$\begin{pmatrix} 1 & 10 & 3 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{pmatrix} \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{pmatrix} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{pmatrix} = \begin{pmatrix} 1 & 10 & 3 \\ 2 & -1 & 3 \\ 1 & 2 & 3 \end{pmatrix}^{-1} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{pmatrix} = \begin{pmatrix} \frac{3}{8} & 1 & -\frac{11}{8} \\ \frac{1}{8} & 0 & -\frac{1}{8} \\ -\frac{5}{24} & -\frac{1}{3} & \frac{7}{8} \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \underline{x}_1 \\ \underline{x}_2 \\ \underline{x}_3 \end{pmatrix} = \begin{pmatrix} -0.625 \\ 0.125 \\ 0.125 \end{pmatrix}$$

We use **Mathematica 5.0** to find the inverse.

Also, the system (3.14) can be written in matrix form as follows,

$$\begin{pmatrix} 9 & 17 & 5 \\ 5 & 2 & 4 \\ 4 & 4 & 6 \end{pmatrix} \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} 9 & 17 & 5 \\ 5 & 2 & 4 \\ 4 & 4 & 6 \end{pmatrix}^{-1} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} \frac{2}{107} & \frac{41}{107} & -\frac{29}{107} \\ \frac{7}{107} & -\frac{17}{107} & \frac{11}{107} \\ -\frac{6}{107} & -\frac{16}{107} & \frac{67}{107} \end{pmatrix} \begin{pmatrix} 3 \\ 1 \\ 1 \end{pmatrix}$$

$$\Rightarrow \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} = \begin{pmatrix} 0.168 \\ 0.140 \\ 0.308 \end{pmatrix}$$

Hence the fuzzy solution of the system (3.11) is

$$\begin{aligned} x_1 &= [\underline{x}_1, \bar{x}_1] = [-0.625, 0.168] \\ x_2 &= [\underline{x}_2, \bar{x}_2] = [0.125, 0.140] \\ x_3 &= [\underline{x}_3, \bar{x}_3] = [0.125, 0.308] \end{aligned}$$

5. Conclusion

In the above paper, we showed the solution technique to solve a fuzzy system of linear equation in matrix form. We discussed two types of application of the method, where the first one is for the fuzzy system and the second one is for the fully fuzzy system. Also, we showed three types of applications namely triangular, trapezoidal and interval. Finally, we showed an application for the fully fuzzy type system. In this paper, we considered only real fuzzy number. Our next goal will be to inspect the complex fuzzy number.

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